Another Derivation of Binary Error Rates as a Function of Signal-to-Noise Power Ratio for Various Modulation Schemes

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Unclassified

The probability of error as a function of signal-to-noise ratio, SNR, is derived for matched filter or coherent cross-correlation receivers, such as coherent PSK, non-coherent two-channel (FSK) type receivers, and differentially coherent phase shift keyed signals.

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THE MATCHED FILTER CASE

The transmitter sends $f(t)$ which is one of two signals $s_1(t)$ or $s_2(t)$ under control of the information. The duration of a signal element is T seconds. The receiver introduces the signal plus noise, $x(t)$, into a filter having impulse response:

$$h(t) = s_1(T-t) - s_2(T-t)$$
$$x(t) = f(t) + n(t).$$

The filter output is sampled and dumped at intervals of T in sync with the signal. The output, R, at the time of sampling, is thus

$$R(T) = \int_0^T x(t)s_1(t)\,dt - \int_0^T x(t)s_2(t)\,dt,$$

and, in a binary symmetric channel, an error is made when $s_1(t)$ is sent and $R < 0$, or when $s_2(t)$ is sent, and $R > 0$.

Clearly, the probability of error, P_e, is, when $s_1(t)$ is sent,

$$P_e = P(R < 0) = P\left[\left(\int_0^T x(t)s_1(t)\,dt - \int_0^T x(t)s_2(t)\,dt\right) < 0\right].$$

Let $\int_0^T s_1^2(t)\,dt = \int_0^T s_2^2(t)\,dt = ST$ (joules);

$\int_0^T s_1(t)s_2(t)\,dt = \rho ST$, where ρ is the cross-correlation coefficient between signals $s_1(t)$ and $s_2(t)$;

$$\rho = -1 \Rightarrow \text{anti-correlated}$$
$$\rho = 0 \Rightarrow \text{orthogonal}$$

If N_0 watts/cycle noise power density is present at the input to the filter, the output noise power, N, is thus

$$N = \frac{N_0}{2\pi}\int_{-\infty}^{\infty} |H(w)|^2 \, dw = \frac{N_0}{T}\int_0^T h^2(t)\, dt = 2S(1-\rho)\, N_0.$$

Since the filter is linear, the noise voltage samples at the output are distributed normally with zero mean and variance σ^2.

$$\sigma = \sqrt{2ST(1-\rho)N_0}\ .$$

Assume that $s_2(t)$ is sent. We have seen the probability of error to be the probability that $R > 0$.

The expected value of R, $E(R)$ is

$$E(R) = E\left\{\int_0^T [s_2(t) + n(t)][s_1(t) - s_2(t)]\right\} = -ST(1-\rho)$$

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$$P_e = P(R > 0) = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left[\frac{R+E(R)}{2\sigma}\right]^2} dR.$$

A simple translation along the R axis, letting $V = \dfrac{R+E(R)}{\sigma}$,

$$P_e = \int_{\frac{E(R)}{\sigma}}^{\infty} e^{-\frac{V^2}{2}} dV,$$

but

$$\frac{E(R)}{\sigma} = \frac{ST(1-\rho)}{\sqrt{2ST\,(1-\rho)N_0}} = \sqrt{\frac{ST(1-\rho)}{2N_0}},$$

and, assuming the bandwidth $W = 2/T$ (including the negative frequencies),

$$P_e = \int_{\sqrt{\frac{S(1-\rho)}{N}}}^{\infty} e^{-\frac{V^2}{2}} dV.$$

This function is plotted for $\rho = 0$ and $\rho = -1$ on Fig. 1, c and a, respectively.

Note should be made of the fact that the equations and results are identical if the matched filters are replaced with active cross-correlators using stored references $s_1(t)$ and $s_2(t)$. This case requires knowledge of the RF phase of the received signal, however. A simple example of such a system is coherent phase shift keying (PSK) in which $s_1(t)$ and $s_2(t)$ are sinusoids differing only by a phase shift of π.

THE FSK CASE

When the received signal is known except for RF phase, as in cross-correlation receivers at bandpass which compare the envelope of each channel, or simple frequency shift keying (FSK), the model can be thought of as sine wave plus gaussian noise in one channel and gaussian noise in the other. For FSK, the signal is one of two different frequencies, and this analysis assumes that the power in the unkeyed channel is independent of signal power in the other channel.

Let $P(V_s)$ be the probability density function of the envelope of the sum of a sine wave $P \sin w_0 t$ and gaussian noise of power N. The signal power, S, is $P^2/2$.

$$P(V_s) = \frac{V_s}{N} \exp\left[-\frac{(V_s^2 + P^2)}{2N}\right] I_0\left(\frac{PV_s}{N}\right) \qquad V_s \geq 0^*$$

In the noise channel, the probability density function of the envelope, V_n, is.

$$P(V_n) = \frac{V_n}{N} \exp\left[-\frac{V_n^2}{2N}\right] \qquad V_n \geq 0.$$

It is assumed that the noise powers in both channels are equal.

The probability, thus, that V_n exceeds arbitrary level β is

$$P(V_n > \beta) = \int_\beta^\infty P(V_n) = \frac{1}{N} \cdot \frac{(-2N)}{2} \int_\beta^\infty e^{-\frac{V_n^2}{2N}} \frac{(-V_n)}{2N} \, dV_n = e^{\frac{-\beta^2}{2N}}$$

The probability of bit error is the probability that the envelope of the noise exceeds the envelope of signal plus noise,

$$P_e = P(V_n > V_s),$$

which is the joint probability that $V_s = \beta$ and $V_n > \beta$ taken over all β.

$$P_e = \int_{\beta=0}^\infty P(V_s = \beta) \, e^{\frac{-\beta^2}{2N}} d\beta$$

$$= \int_{\beta=0}^\infty \left[\frac{\beta}{N} e^{-\left[\frac{\beta^2 + P^2}{2N}\right]} I_0\left(\frac{P\beta}{N}\right)\right] e^{\frac{-\beta^2}{2N}} d\beta$$

$$= \frac{e^{-\frac{P^2}{2N}}}{N} \int_0^\infty \beta \, e^{-\frac{\beta^2}{N}} I_0\left(\frac{P\beta}{N}\right) d\beta.$$

Making some algebraic manipulations,

$$\text{Let } N = 2V$$
$$P = 2Q;$$

* W. B. Davenport and W. L. Root, *An Introduction to the Theory of Random Signals and Noise*, McGraw-Hill, New York, 1958, p. 166.

$$P_e = \frac{1}{2V} e^{\left(-\frac{4Q^2}{4V}+\frac{Q^2}{2V}\right)} \underbrace{\int_0^\infty \frac{\beta}{V} e^{-\frac{\beta^2}{2V}-\frac{Q^2}{2V}} I_0\left[\frac{2Q\beta}{2V}\right] d\beta}_{=1^*}$$

$$P_e = \frac{1}{2} e^{-\frac{Q^2}{2V}} = \frac{1}{2} e^{-\frac{P^2/4}{N}}$$

$$P_e = \frac{1}{2} e^{-\frac{S}{2N}}.$$

This function is plotted on Fig. 1, curve d.

THE DIFFERENTIAL PHASE SHIFT KEY CASE.

If the channel characteristics are slowly varying with respect to keying rate, the RF phase of a previous bit can be used as a phase reference to the baseband. The input is passed down a delay line one bit long and then multiplied by the input. The sign of this product (averaged over one information period, T) is the information bit.

Let the signal, $s(t)$, be of the form

$$s(t) = m(t) \cos w_0 t$$

where $m(t)$ equals plus or minus P under control of the information, and the noise,

$$n(t) = x(t) \cos w_0 t + y(t) \sin w_0 t$$

where x and y are normal random variables with zero mean.

$$S = \text{signal power} = E\left[s^2(t)\right] = \frac{P^2}{2} \text{ (watts)}$$

$$N = \text{noise power} = E\left[n^2(t)\right] = \frac{1}{2}\left[\sigma_x^2 + \sigma_y^2\right] = \sigma^2 .$$

The input to the receiver is: $s(t) + n(t)$; the delayed reference then becomes:

$$s(t-T) + n(t-T).$$

Assuming the delay T and bandwidth W of the system are chosen properly, $n(t-T)$ is independent of $n(t)$ and can be written as $u(t) \cos w_0 t + v(t) \sin w_0 t$, where u and v are normal random variables with zero mean. The signal term in the reference channel, $m(t-T)$ equals

* cf. Table of Q Functions.

$m(t)$ if mark is sent, and equals $-m(t)$ if space is sent. Assuming consecutive marks are sent, the inputs to the receiver-multiplier are:

$$[x(t) + m(t)] \cos w_0 t + y(t) \sin w_0 t$$

$$[u(t) + m(t)] \cos w_0 t + v(t) \sin w_0 t \,.$$

Neglecting the factor of $1/2$, the lower sideband of the output of the multiplier, Z, is, dropping the independent variable, t,

$$Z = (x + m) \cdot (u + m) + yv.$$

[redacted]

In a binary symmetric channel the probability of error, P_e, is thus the probability that Z is less than zero.

$$P_e = P(Z < 0).$$

After some algebraic manipulation, Z can be written

$$\left[\left(\frac{x + u + 2m}{2}\right)^2 + \left(\frac{y + v}{2}\right)^2\right] - \left[\left(\frac{x - u}{2}\right)^2 + \left(\frac{y - v}{2}\right)^2\right]^*$$

$$P(Z < 0) = P\left[\left(\frac{x - u}{2}\right)^2 + \left(\frac{y - v}{2}\right)^2 > \left(\frac{x + u + 2m}{2}\right)^2 + \left(\frac{y + v}{2}\right)^2\right]$$

$$E\left(\frac{x - u}{2}\right) = E\left(\frac{y - v}{2}\right) = 0$$

$$E\left(\frac{x - u}{2}\right)^2 = \frac{1}{4}\left[E(x)^2 - 2E(xu) + E(u^2)\right] = \frac{N}{2} = \psi \,.$$

Likewise,

$$E\left(\frac{y - v}{2}\right)^2 = \psi \,.$$

Let $R^2 = \left(\frac{x - u}{2}\right)^2 + \left(\frac{y - v}{2}\right)^2$. The Rayleigh density function can be written

$$\phi(R) = \frac{R}{\psi} \exp\left(\frac{-R^2}{2\psi}\right)$$

because $(x - u)/2$ and $(y - v)/2$ are gaussian random variables of zero mean and equal variance.

* Research Report No. 315, ARL—Sylvania.

The probability that an error is made is the probability that R^2 exceeds $g^2 + h^2$ where

$$g = \frac{x + u + 2m}{2}$$

$$h = \frac{y + v}{2}$$

taken over all x, y, u and v. For a particular g and h, $g_1 + h_1$

$$P_{g_1h_1}(Z < 0) = P[R^2 > g_1{}^2 + h_1{}^2] = \int_{\sqrt{g_1{}^2 + h_1{}^2}}^{\infty} \phi(R)\, dR$$

(NOTE: Taking square root of both sides of inequality.)

[redacted]

$$= \int_{\sqrt{g_1{}^2 + h_1{}^2}}^{\infty} \frac{R}{\psi} \exp\left[-\frac{R^2}{2\psi}\right] dR$$

$$= \exp\left[-\frac{(g_1{}^2 + h_1{}^2)}{2\psi}\right].$$

Then the probability over all g and h of error $P(Z < 0)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\phi(g_1h_1)}_{\substack{\text{Probability of}\\ \text{event (Rayleigh}\\ \text{distributed)}}} \underbrace{P_{g_1h_1}(Z < 0)}_{\substack{\text{Conditional probability}\\ \text{of error, given event}}}\, dg\, dh$$

$$P_e = \int_{g_1 = -\infty}^{\infty}\int_{h_1 = -\infty}^{\infty} \frac{1}{2\pi\psi} \exp\left\{-\frac{[g_1 - E(g_1)]^2 + [h_1 - E(h_1)]^2}{2\psi}\right\}$$

$$\cdot \exp\left\{-\frac{(g_1{}^2 + h_1{}^2)}{2\psi}\right\} dg\, dh.$$

Fortunately, this integral is evaluated on p. 105 of Burington and May's *Handbook of Probability and Statistics*.

$$P_e = \frac{1}{2} \exp -\left\{\frac{E^2(g) + E^2(h)}{4\psi}\right\},$$

but $E^2(g) = m^2$, and $E^2(h) = 0$

$$P_e = \frac{1}{2}\exp\left\{-\frac{m^2}{4\psi}\right\} = \frac{1}{2}\exp\left\{-\frac{m^2}{2}\cdot\frac{1}{2\psi}\right\};$$

$$P_e = \frac{1}{2}\exp\left\{-\frac{S}{N}\right\}.$$

This function is plotted in Fig. 1, curve *b*.

Comparison of this result with FSK shows that the significant difference between them in a constant S/N_0 environment is the bandwidth saving using DPSK.

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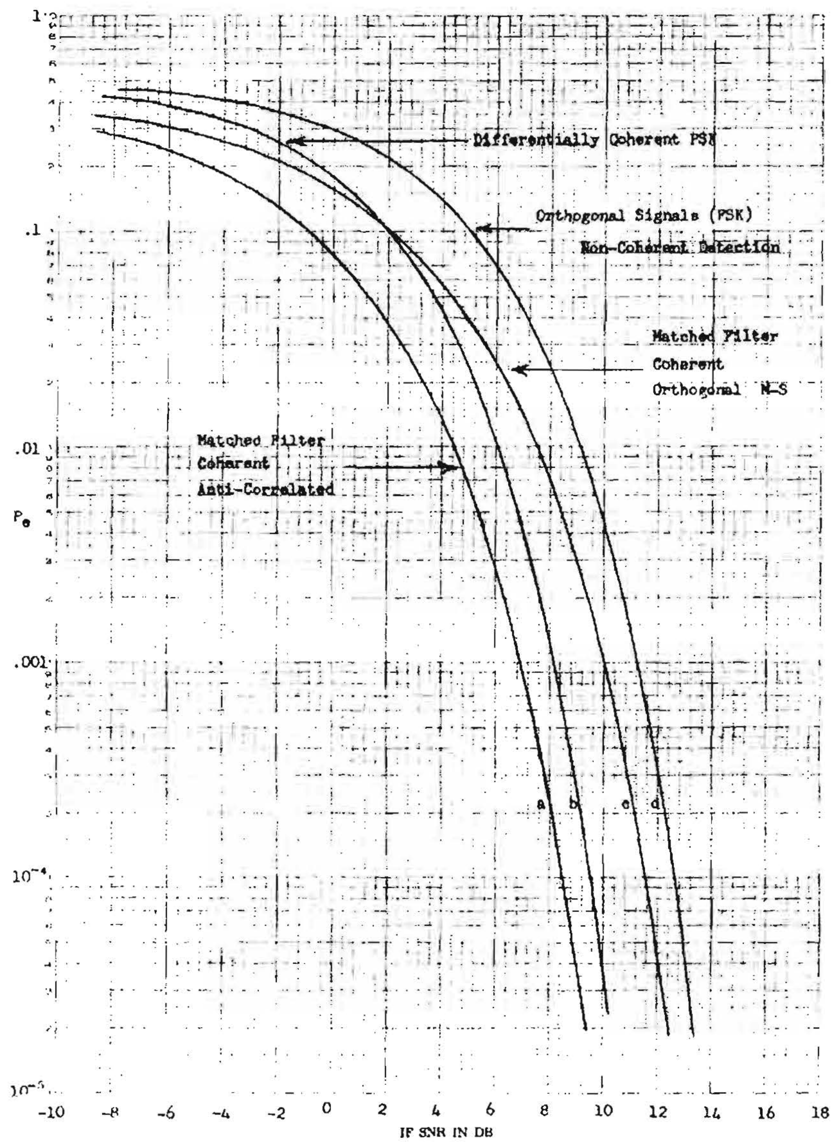
Differentially Coherent PSK
Orthogonal Signals (FSK)
Non-Coherent Detection
Matched Filter
Coherent
Orthogonal N-S
Matched Filter
Coherent
Anti-Correlated
a
b
c
d
1
.1
.01
P_e
.001
10^{-4}
10^{-5}
-10
-8
-6
-4
-2
0
2
4
6
8
10
12
14
16
18
IF SNR IN DB

Fig. 1